

# Spectral properties of Schrödinger-type operators and large-time behavior of the solutions to the corresponding wave equation

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**Abstract.** Let  $L$  be a linear, closed, densely defined in a Hilbert space operator, not necessarily selfadjoint.

Consider the corresponding wave equations

$$\begin{aligned} (1) \quad & \ddot{w} + Lw = 0, \quad w(0) = 0, \quad \dot{w}(0) = f, \quad \dot{w} = \frac{dw}{dt}, \quad f \in H. \\ (2) \quad & \ddot{u} + Lu = fe^{-ikt}, \quad u(0) = 0, \quad \dot{u}(0) = 0, \end{aligned}$$

where  $k > 0$  is a constant. Necessary and sufficient conditions are given for the operator  $L$  not to have eigenvalues in the half-plane  $\operatorname{Re} z < 0$  and not to have a positive eigenvalue at a given point  $k_a^2 > 0$ . These conditions are given in terms of the large-time behavior of the solutions to problem (1) for generic  $f$ .

Sufficient conditions are given for the validity of a version of the limiting amplitude principle for the operator  $L$ .

A relation between the limiting amplitude principle and the limiting absorption principle is established.

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## 1. Introduction

Let  $L$  be a linear, densely defined, closed operator in a Hilbert space  $H$ . Our results and techniques are valid in a Banach space also, but we wish to think about  $L$  as of a Schrödinger-type operator in a Hilbert space and, at times, think that  $L$  is selfadjoint. For a Schrödinger operator  $L = -\nabla^2 + q(x)$  the resolvent  $(L - k^2)^{-1}$ ,  $\operatorname{Im} k > 0$ , is an integral operator with a kernel  $G(x, y, k)$ , its resolvent kernel. If  $q$  is a real-valued function, sufficiently rapidly decaying then  $L$  is selfadjoint,  $G(x, y, k)$  is analytic with respect to  $k$  in the half-plane  $\operatorname{Im} k > 0$ , except, possibly, for a finitely many simple poles  $ik_j$ ,  $k_j > 0$ , the

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semiaxis  $k \geq 0$  is filled with the points of absolutely continuous spectrum of  $L$ , and there exists a limit

$$\lim_{\epsilon \rightarrow 0} G(x, y, k + i\epsilon) = G(x, y, k)$$

for all  $k > 0$ .

Sufficient conditions for  $k^2 = 0$  not to be an eigenvalue of  $L$  are found in papers [5], [6]. Spectral analysis of the Schrödinger operators is presented in many books (see, for example, [2] and [11]). In papers [3], [4], such an analysis was given in a class of domains with infinite boundaries apparently for the first time, see also [8]. In [7] an eigenfunctions expansion theorem was proved for non-selfadjoint Schrödinger operators with exponentially decaying complex-valued potential  $q$ . The operator  $L$  in this paper is not necessarily assumed to be selfadjoint.

In [1] the validity of the limiting amplitude principle for some class of selfadjoint operators  $L$  has been established.

This principle says that, as  $t \rightarrow \infty$ , the solution to problem

$$\ddot{u} + Lu = fe^{-ikt}, \quad u(0) = 0, \quad \dot{u}(0) = 0, \quad \dot{u} = \frac{du}{dt}, \quad (1.1)$$

has the following asymptotics

$$u = e^{-ikt}v + o(1), \quad t \rightarrow \infty, \quad (1.2)$$

where  $k$  is a real number and  $v \in H$  solves the equation

$$Lv - k^2v = f. \quad (1.3)$$

The  $v$  is called *the limiting amplitude*. It turns out that a more natural definition of the limiting amplitude is:

$$v = \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t u(s) e^{iks} ds, \quad (1.4)$$

if this limit exists and solves equation (1.3).

Why is this definition more natural than (1.2)? There are good reasons for this. One of the reasons is: if (1.2) and (1.3) hold, then the limit (1.4) exists and solves equation (1.3). The other reason is: the limit (1.4) may exist and solve equation (1.3) although the limit (1.2) does not exist.

**Example.** If  $u = e^{ikt}v + e^{ik_1t}v_1$ , then the limit (1.2) does not exist, while the limit (1.4) does exist and is equal to  $v$ .

To describe our assumptions and results, some preparation is needed.

Consider the problem

$$\ddot{w} + Lw = 0, \quad w(0) = 0, \quad \dot{w}(0) = f. \quad (1.5)$$

Assuming that  $\|u(t)\| \leq ce^{at}$ , where  $c > 0$  stands throughout the paper for various generic constants, and  $a \geq 0$  is a constant, one can define the Laplace transform of  $u(t)$ ,

$$\mathcal{U} := \mathcal{U}(p) := \int_0^\infty e^{-pt} u(t) dt, \quad \sigma > a,$$

where  $p = \sigma + i\tau$ ,  $\text{Re } p = \sigma$ .

Let us take the Laplace transform of (1.1) and of (1.5) to get

$$L\mathcal{U} + p^2\mathcal{U} = \frac{f}{p + ik}, \quad (1.6)$$

and

$$L\mathcal{W} + p^2\mathcal{W} = f, \quad (1.7)$$

where

$$\mathcal{W} = \mathcal{W}(p) = \int_0^\infty w(t)e^{-pt}dt.$$

We also denote  $\mathcal{W}(p) := \bar{w}(t)$ .

The complex plane  $p$  is related to the complex plane  $k$  by the formula

$$p = -ik, \quad k = k_1 + ik_2, \quad k_2 \geq 0, \quad \sigma = k_2 \geq 0. \quad (1.8)$$

We assume throughout that  $f$  is generic in the following sense:

*If  $I$  is the identity operator and a point  $p$  is a pole of the kernel of the operator  $(L + p^2I)^{-1}$ , then it is a pole of the same order of the element  $(L + p^2I)^{-1}f = \mathcal{W}$ .*

If  $k^2$  is an eigenvalue of  $L$  and  $\operatorname{Re} k^2 < 0$ , then  $\operatorname{Im} k > 0$ , where  $k = |k|e^{\frac{i \arg k^2}{2}}$ ,  $p = -ik$ , so  $\sigma = \operatorname{Re} p > 0$ . Let  $k > 0$  and assume that  $-k^2 < 0$  is an eigenvalue of  $L$ . Then  $ik$  is a pole of the resolvent kernel  $G(x, y, k)$ , and  $p = -i(ik) = k$  is a pole of the kernel of the operator  $(L + p^2I)^{-1}$ . If  $k^2 > 0$  is an eigenvalue of  $L$ , then  $p = -ik$  is a pole of the operator  $(L + p^2I)^{-1}$ .

The following known facts from the theory of Laplace transform will be used.

**Proposition 1.1.** *An analytic in the half-plane  $\sigma > \sigma_0 \geq 0$  function  $F(p)$  is the Laplace transform of a function  $f(t)$ , such that  $f(t) = 0$  for  $t < 0$  and*

$$\int_0^\infty |f(t)|^2 e^{-2\sigma_0 t} dt < \infty \quad (1.9)$$

*if and only if*

$$\sup_{\sigma > \sigma_0} \int_{-\infty}^\infty |F(\sigma + i\tau)|^2 d\tau < \infty. \quad (1.10)$$

**Proposition 1.2.** *If  $F(p) = \overline{f(t)}$ , then*

$$\frac{F(p)}{p} = \overline{\int_0^t f(s)ds}. \quad (1.11)$$

Let us now formulate the main Assumptions A and B standing throughout this paper.

**Assumption A.** For a generic  $f$  the  $\mathcal{W}(p) = (L + p^2)^{-1}f$  is analytic in the half-plane  $\sigma > 0$ , except, possibly, at a finitely many simple poles at the points  $-ik_j$ ,  $1 \leq j \leq J$ ,  $k_j$  are real numbers, and at the points  $\kappa_m$ ,  $\operatorname{Re} \kappa_m > 0$ ,

$$\mathcal{W}(p) = \sum_{j=1}^J \frac{v_j}{p + ik_j} + \mathcal{W}_1(p) + \sum_{m=1}^M \frac{b_m}{p - \kappa_m}, \quad (1.12)$$

where  $v_j$  and  $b_m$  are some elements of  $H$ ,  $\mathcal{W}_1(p)$  is an analytic function in the half-plane  $\operatorname{Re} p = \sigma > 0$ , continuous up to the imaginary axis  $\sigma = 0$ , and satisfying the following estimate

$$\|\mathcal{W}_1(p)\| \leq \frac{c}{1 + |p|^\gamma}, \quad \gamma > \frac{1}{2}. \quad (1.13)$$

**Assumption B.** There exists the limit

$$\lim_{\sigma \rightarrow 0} \|\mathcal{W}_1(\sigma - ik) - \mathcal{W}_1(-ik)\| = 0 \quad (1.14)$$

for all real numbers  $k$ .

**Theorem 1.3.** *Let the Assumption A hold. Then a necessary and sufficient condition for the operator  $L$  to have no eigenvalues in the half-plane  $\operatorname{Re} k^2 < 0$  is the validity of the estimate*

$$\left\| \int_0^t w(s) ds \right\| = O(e^{\epsilon t}), \quad t \rightarrow \infty, \quad (1.15)$$

for an arbitrary small  $\epsilon > 0$ .

A necessary and sufficient condition for the operator  $L$  not to have any positive eigenvalues  $k^2 > 0$  is the validity of the estimate

$$\left\| \frac{1}{t} \int_0^t e^{iks} w(s) ds \right\| = o(1), \quad t \rightarrow \infty, \quad \forall k \in \mathbb{R}. \quad (1.16)$$

A point  $ik_0 > 0, k_0 > 0$ , is not a pole of the resolvent kernel of the operator  $(L - k^2 - i0)^{-1}$  if and only if estimate (1.16) holds with  $k = k_0 > 0$ .

**Remark.** If condition (1.16) holds for  $k = 0$ , then  $\left\| \int_0^t w(s) ds \right\| = o(t)$ , so condition (1.15) holds, and the operator  $L$  has no eigenvalues in the half-plane  $\operatorname{Re} k^2 < 0$ .

**Theorem 1.4.** *Let the Assumptions A and B hold. Suppose that estimates (1.14) and (1.15) hold. Then the limiting amplitude principle (1.4) holds for every  $k \in \mathbb{R}, k \neq k_j, 1 \leq j \leq J$ .*

In section 2, proofs are given.

## 2. Proofs

### 2.1. Proof of Theorem 1.3

From the Assumption A and Proposition 1.1, it follows that  $\mathcal{W}(p)$  is a Laplace transform of a function  $w(t)$  such that

$$w(t) = \sum_{j=1}^J v_j e^{-ik_j t} + \sum_{m=1}^M b_m e^{\kappa_m t} + w_1(t), \quad (2.1)$$

where

$$w_1(t) = \frac{1}{2\pi i} \int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} e^{pt} \mathcal{W}_1(p) dp, \quad (2.2)$$

and the integral in (2.2) converges in  $L^2$ -sense due to the assumption (1.13). It is clear from formula (2.1) that all  $b_m = 0$  if and only if estimate (1.15) holds with  $0 < \epsilon < \min_{1 \leq m \leq M} \operatorname{Re} \kappa_m$ . This proves the first conclusion of Theorem 1.3.

Let us calculate the expression on the left side of formula (1.16) and show that this expression is  $o(1)$  unless  $k = k_j$  for some  $1 \leq j \leq J$ . In this calculation it is assumed that  $L$  does not have any eigenvalues in the half-plane  $\operatorname{Re} k^2 < 0$ , in other words, that all  $b_m = 0$ . Otherwise the expression on the left of formula (1.16) tends to infinity as  $t \rightarrow \infty$  at an exponential rate.

If all  $b_m = 0$  in (2.1), then

$$\sum_{j=1}^J v_j \frac{1}{t} \int_0^t e^{i(k-k_j)t} dt + \frac{1}{t} \int_0^t w_1(t) e^{ikt} dt := I_1 + I_2. \quad (2.3)$$

If  $k$  and  $k_j$  are real numbers, then

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t e^{i(k-k_j)t} dt = \begin{cases} 1, & k = k_j, \\ 0, & k \neq k_j. \end{cases} \quad (2.4)$$

Thus,  $I_1 = 0$  if and only if  $k$  does not coincide with any of  $k_j$ ,  $1 \leq j \leq J$ .

Let us prove that

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t w_1(t) e^{ikt} dt = 0. \quad (2.5)$$

By proposition (1.2) and the Mellin inversion formula, one has

$$I := \frac{1}{t} \int_0^t w_1(t) e^{ikt} dt = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \mathcal{W}_1(p-ik) \frac{e^{pt}}{pt} dp, \quad (2.6)$$

where  $\text{Re } p = \sigma > 0$  can be chosen arbitrarily small.

Let  $pt = q$ , take  $\sigma = \frac{1}{t}$ , write  $q = 1 + is$ , and write the integral on the right side of (2.6) as:

$$I = \frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} \mathcal{W}_1\left(\frac{q}{t} - ik\right) \frac{q}{t} \frac{e^q}{q^2} dq. \quad (2.7)$$

If one uses estimate (1.13) and formula  $|q| = (1 + s^2)^{1/2}$ , then one obtains the following inequality

$$||I|| \leq \frac{1}{2\pi t} \int_{-\infty}^{\infty} \frac{1}{(1 + s^2)^{1/2}} \frac{cds}{[1 + |\frac{1+is}{t} - ik|\gamma]} = \frac{c}{2\pi t^{1-\gamma}} \int_{-\infty}^{\infty} \frac{1}{(1 + s^2)^{1/2}} \frac{ds}{(t^\gamma + [1 + (s - kt)^2]^{\gamma/2})}. \quad (2.8)$$

Let  $s = ty$ . Then the integral on the right side of (2.8) can be written as

$$\begin{aligned} & \frac{ct}{2\pi t^{1-\gamma}} \int_{-\infty}^{\infty} \frac{dy}{(1 + t^2 y^2)^{1/2}} \frac{1}{(t^\gamma + [1 + t^2(y - k)^2]^{\gamma/2})} \\ &= \frac{c}{2\pi} \int_{-\infty}^{\infty} \frac{dy}{(1 + t^2 y^2)^{1/2}} \frac{1}{(1 + [t^{-2} + (y - k)^2]^{\gamma/2})} \\ &\leq \frac{c}{2\pi} \int_{-\infty}^{\infty} \frac{dy}{(1 + t^2 y^2)^{1/2}} \frac{1}{[1 + (y - k)^\gamma]} \rightarrow 0, \text{ as } t \rightarrow \infty, \end{aligned} \quad (2.9)$$

and the convergence of the last integral to zero is uniform with respect to  $k \in \mathbb{R}$ .

Thus

$$\lim_{t \rightarrow \infty} ||I|| = 0. \quad (2.10)$$

From (2.3)-(2.5) the last two conclusions of Theorem 1.3 follow. Theorem 1.3 is proved.  $\square$

## 2.2. Proof of Theorem 1.4

Using Proposition 1.2, equation (1.6), and the Mellin formula, one gets

$$\frac{1}{t} \int_0^t u(t) e^{ikt} dt = \frac{1}{t} \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{\mathcal{U}(p-ik)}{p} e^{pt} dp, \quad (2.11)$$

where, according to (1.6),

$$\mathcal{U}(p-ik) = \frac{\mathcal{W}(p-ik)}{p}. \quad (2.12)$$

Let  $\sigma = \frac{1}{t}$  and  $pt = q$ . Then

$$\frac{1}{t} \int_0^t u(t) e^{ikt} dt = \frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} \mathcal{W}\left(\frac{q}{t} - ik\right) \frac{e^q}{q^2} dq. \quad (2.13)$$

Estimate (1.15) and Theorem 1.3 imply that all  $b_m = 0$  in formula (2.1). Therefore, using formula (2.1) with  $b_m = 0$ , one gets

$$\mathcal{W} = \sum_{j=1}^J v_j \frac{1}{p + ik_j} + \mathcal{W}_1,$$

and

$$\mathcal{W}\left(\frac{q}{t} - ik\right) = \mathcal{W}_1\left(\frac{q}{t} - ik\right) + \sum_{j=1}^J v_j \frac{1}{\frac{q}{t} - i(k - k_j)}. \quad (2.14)$$

One has  $\overline{t^n} = \frac{n!}{p^{n+1}}$ . Therefore

$$\frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} \frac{e^q}{q^2} dq = 1,$$

and

$$\lim_{t \rightarrow \infty} \frac{v_j}{2\pi i} \int_{1-i\infty}^{1+i\infty} \frac{1}{\frac{q}{t} - i(k - k_j)} \frac{e^q}{q^2} dq = \begin{cases} \frac{iv_j}{k - k_j}, & k \neq k_j, \\ \infty, & k = k_j. \end{cases} \quad (2.15)$$

Furthermore,

$$\lim_{t \rightarrow \infty} \frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} \mathcal{W}_1\left(\frac{q}{t} - ik\right) \frac{e^q}{q^2} dq = \mathcal{W}_1(-ik), \quad (2.16)$$

as follows from assumption (1.14) and the Lebesgue's dominated convergence theorem if one passes to the limit  $t \rightarrow \infty$  under the sign of the integral (2.16). Let us check that this  $v$  solves equation (1.3). This would conclude the proof of Theorem 1.4. We need a lemma.

**Lemma 1.** *If  $h \in L^1_{loc}(0, \infty)$  and the limit  $\lim_{t \rightarrow \infty} t^{-1} \int_0^t h(s) ds$  exists, then the limit  $\lim_{p \rightarrow 0} p \int_0^\infty e^{-pt} h(t) dt$  exists, and*

$$\lim_{t \rightarrow \infty} t^{-1} \int_0^t h(s) ds = \lim_{p \rightarrow 0} p \int_0^\infty e^{-pt} h(t) dt. \quad (2.17)$$

**Proof of Lemma 1.** One has

$$p \int_0^\infty e^{-pt} h(t) dt = p e^{-pt} \int_0^t h(s) ds|_0^\infty + p^2 \int_0^\infty t e^{-pt} t^{-1} \int_0^t h(s) ds dt.$$

For any  $p > 0$  one has

$$p e^{-pt} \int_0^t h(s) ds|_0^\infty = 0.$$

Let  $q = pt$  and denote  $H(t) := t^{-1} \int_0^t h(s) ds$ ,  $J := \lim_{t \rightarrow \infty} H(t)$ . Then

$$\lim_{p \rightarrow 0} p^2 \int_0^\infty t e^{-pt} t^{-1} \int_0^t h(s) ds dt = \lim_{p \rightarrow 0} \int_0^\infty q e^{-q} H(qp^{-1}) dq.$$

Passing in the last integral to the limit  $p \rightarrow 0$  one obtains (2.17). Lemma 1 is proved.  $\square$

Using equation (2.17), one writes  $v = \lim_{p \rightarrow 0} p \mathcal{U}(p - ik)$ , where  $\mathcal{U}$  solves equation (1.6). Thus,

$$L\mathcal{U}(p - ik) + (p - ik)^2 \mathcal{U}(p - ik) = p^{-1} f.$$

Multiplying both sides of this equation by  $p$  and passing to the limit  $p \rightarrow 0$ , one obtains equation (1.3). In the passage to the limit under the sign of the unbounded operator  $L$  the assumption that  $L$  is closed was used.

Thus, the conclusion of Theorem 1.4 follows.  $\square$

If the limit (1.14) exists at a point  $p = i\tau$  then one says that the limiting absorption principle holds for the operator  $L$  at the point  $k = ip = i(-ik) = k, k > 0$ .

Thus, Assumption B means that the limiting absorption principle holds for  $L$  at the point  $k > 0$ , that is,  $\lim_{\epsilon \rightarrow 0} (L - k^2 - i\epsilon)^{-1} f$  exists.

### 3. Applications

Let  $L = -\nabla^2 + q(x)$ , where  $q(x)$  is a real-valued function,  $|q(x)| \leq c(1 + |x|)^{-2-\epsilon}$ ,  $\epsilon > 0$ ,  $x \in \mathbb{R}^3$ . Then  $L$  is selfadjoint on the domain  $H^2(\mathbb{R}^3)$ . Its resolvent  $(L - k^2 - i0)^{-1}$  satisfies Assumptions A and B if one keeps in mind the following.

Let  $G(x, y, k)$  be the resolvent kernel of  $L$ , that is, the kernel of the operator  $(L - k^2 - i0)^{-1}$ ,

$$LG(x, y, k) = -\delta(x - y) \quad \text{in } \mathbb{R}^3,$$

$G \in L^2(\mathbb{R}^3)$  for  $\text{Im } k > 0$ . If  $f \in L^2(\mathbb{R}^3)$  is compactly supported, then for  $k > 0$  the function

$$v(x) := (L - k^2 - i0)^{-1}f = \int_{\mathbb{R}^3} G(x, y, k)f(y)dy$$

does not necessarily belong to  $L^2(\mathbb{R}^3)$ .

For example, if  $q(x) = 0$ , then  $G(x, y, k) = \frac{e^{ik|x-y|}}{4\pi|x-y|}$ , and the function

$$v(x, k) = \int_{|y| \leq 1} g(x, y, k)dy = O\left(\frac{1}{|x|}\right) \quad (3.1)$$

does not belong to  $L^2(\mathbb{R}^3)$  (except for those  $k > 0$  for which  $x(x, k) = 0$  in the region  $|y| \geq 1$ . These numbers  $k > 0$  are the zeros of the Fourier transform of the characteristic function of the ball  $|y| \leq 1$ , see [10], Chapter 11.

By this reason the abstract results of theorem (1.3) and (1.4) can be used in applications if one defines some subspace of  $H$ , for example, a subspace of functions with compact support, denote by  $\mathcal{P}$ , a projection operator on this subspace, and replaces  $\mathcal{W}$  and  $\mathcal{W}_1$  by  $\mathcal{P}\mathcal{W}$  and  $\mathcal{P}\mathcal{W}_1$  in equations (1.12) and (1.14). For example, the function (3.1) one replaces by  $\eta(x)v(x, k)$ , where  $\eta(x)$  is a characteristic function of a compact subset of  $\mathbb{R}^3$ .

The analytic properties of  $\eta(x)v(x, k)$  and of  $v(x, k)$  as functions of  $k$  are the same. A similar suggestion is used in [1].

With the above in mind, one knows (for example, from [2] or [11]) that Assumptions A and B hold for  $L = -\nabla^2 + q(x)$ .

Consequently, the conclusions of Theorems 1.3 and 1.4 hold.

In addition, the assumptions

$$|q(x)| \leq c(1 + |x|)^{-2-\epsilon}, \quad \epsilon > 0, \quad \text{Im } q = 0,$$

imply that  $L$  does not have positive eigenvalues, so all  $v_j = 0$ , and zero is not an eigenvalue of  $L \geq 0$  if  $\epsilon > 0$  (see [5], [6]).

A new method for estimating of large time behavior of solutions to abstract evolution problems is developed in [9], where some applications of this method are given.

### References

- [1] D. Eidus. *The limiting amplitude principle*. Russ. Math. Surveys, 24, (1969), 51–94
- [2] D. Pearson. *Quantum scattering and spectral theory*. Acad. Press, London, 1988.
- [3] A.G. Ramm. *Spectral properties of the Schrödinger operator in some domains with infinite boundaries*. Doklady Acad. of Sci. USSR, 152, (1963) 282–285.
- [4] A.G. Ramm. *Spectral properties of the Schrödinger operator in some infinite domains*. Mat. Sborn., 66, (1965), 321–343.
- [5] A.G. Ramm. *Sufficient conditions for zero not to be an eigenvalue of the Schrödinger operator*. J. Math. Phys., 28, (1987), 1341–1343.
- [6] A.G. Ramm. *Conditions for zero not to be an eigenvalue of the Schrödinger operator*. J. Math. Phys. 29, (1988), 1431–1432.
- [7] A.G. Ramm. *Eigenfunction expansion for nonselfadjoint Schrödinger operator*. Doklady Acad. Sci. USSR. 191, (1970), 50–53.

- [8] A.G. Ramm. Scattering by obstacles. D.Reidel, Dordrecht, 1986.
- [9] A.G. Ramm. *Stability of solutions to some evolution problems*. Chaotic Modeling and Simulation (CMSIM), 1, (2011), 17-27.
- [10] A.G. Ramm. Inverse problems. Springer, New York, 2005.
- [11] M. Schechter. Operator methods in quantum mechanics. North Holland, New York, 1981.